

**FROISSART BOUNDS  
FOR AMPLITUDES AND CROSS SECTIONS  
AT HIGH ENERGIES <sup>1</sup>**

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**A b s t r a c t**

High-energy behavior of total cross sections is discussed in experiment and theory. Origin and meaning of the Froissart bounds are described and explained. Violation of the familiar log-squared bound appears to not violate unitarity (contrary to the common opinion), but correspond to rapid high-energy increase of the amplitude in nonphysical regions.

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The elementary particle physics (or, the same, high energy physics) is considered as a separate branch of physics since 1956, when the Rochester University, USA, organized the Conference on High Energy Physics (since then the “Rochester” Conferences gathered once a year in different cities and countries; after 1964 they are biennials called “International Conferences on High Energy Physics”). But, sure, elementary particles and their interactions had been investigated even before 1956. It became clear as early as in '30-ies that particles have interactions of several different kinds. And it was discovered in '40-ies that strong interactions with increasing energy provide increasing multiple meson production. In other words, the role of inelastic processes grows with growing energy in collisions of strong-interacting particles (they are called “hadrons” since 1962, according to suggestion of L.B. Okun).

To 1960, the idea had been formed that scattering of hadrons at very high energies should be similar to the classical diffraction of light on a black (completely absorbing) disc of a finite radius. If this were true, the total interaction cross sections at very high energies should be asymptotically constant, and the elastic cross sections should be a fixed part of the total ones. Angular distribution (or distribution in the momentum transfer  $-t$ ) should look as the diffraction peak  $\sim \exp(bt)$  with a constant slope  $b$ , which is proportional to the radius squared of the disc. Experimental data of those years (in the then available energy interval) seemed to agree with such expectations.

However, in 1961 there appeared two theoretical papers which cast doubts on applicability of such a simple picture. One of them was presented by V.N. Gribov [1]. It showed that the classical diffraction is incompatible with the analytical properties of hadron amplitudes when combined with the cross-channel unitarity condition. This result has become an impetus to construct the Reggeon theory, according to which the diffraction peak changes (shrinks) with increasing energy, even if the total cross section is asymptotically constant.

The other paper was published by Marcel Froissart [2]. Froissart (he is, by the way, a member of the old noble French family) began his work with the hypothesis that total cross sections of hadron interactions may infinitely grow with energy (though no experimental evidence for such possibility had been seen to that time). Then he applied the unitarity condition together with the analyticity of an elastic amplitude, as expressed by the dispersion relations with a finite number of subtractions. Based on such, seemingly very “soft”, conditions (nearly from nothing) Froissart was able to receive quite tangible restriction for a possible energy growth rate of the forward (backward) scat-

tering amplitude, and even stronger restrictions for the fixed angle nonforward (nonbackward) scattering. Since the unitarity condition (the optical theorem) relates the forward elastic amplitude with the total interaction cross section, it appeared that the total cross sections might not grow faster than the logarithm squared of the energy. This result, known as “the Froissart theorem”, has become one of key points when constructing theoretical models for high-energy strong interactions. Moreover, it became a sincere belief for public opinion of the high-energy physics community, that violation of the Froissart theorem would mean violation of unitarity.

In the years after 1961, our knowledge of strong interactions has been significantly expanded to higher energies. The following experimental facts have been definitely established.

- The diffraction peaks shrink indeed with growing energy; their slopes in respect to the momentum transfer grow at least as the logarithm of energy.
- The total cross sections, as is clear now, indeed increase with energy. Existing data for different hadrons agree with the hypothesis that the total cross sections asymptotically grow as  $\ln^2 s$  ( $s$  is the c.m.s. energy squared).

Most advanced in the energy scale are investigations of nucleon-(anti)nucleon interactions, especially if one adds data from cosmic ray studies. Existing values for the total  $pp$  and  $p\bar{p}$  cross sections may be quite satisfactorily described by the curves shown in Fig.1 (taken from Ref.[3]). Their high-energy behavior is proportional to the log-squared energy.

However, the accelerator energy interval available in the pre-LHC era is rather narrow in the logarithmic scale, while data extracted from cosmic ray experiments have great uncertainties. As a result, significant ambiguities may (and do) appear in the description of the data. In particular, possible are “heretic” descriptions, which contradict to the canonically understood Froissart result. For example, Fig.2 (taken from Ref.[4]) shows such a description of the total cross sections which corresponds to the power increase with energy as  $s^\delta$ , though with a small exponent  $\delta \approx 0.08$ . Thus, experiments have not allowed yet to reach a definite conclusion, whether the log-squared energy asymptotics is true or not.

LHC extends the accelerator energies to the values which have been available earlier, but only in cosmic rays. Meanwhile, the accelerator measurements

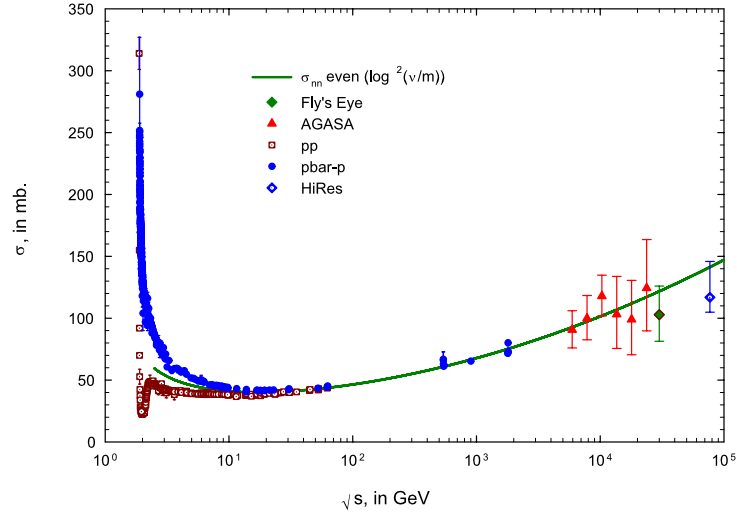


Figure 1: Fit for all data on the total  $pp$  and  $p\bar{p}$  cross sections available before LHC [3]. The curves asymptotically grow as  $\ln^2 s$ .

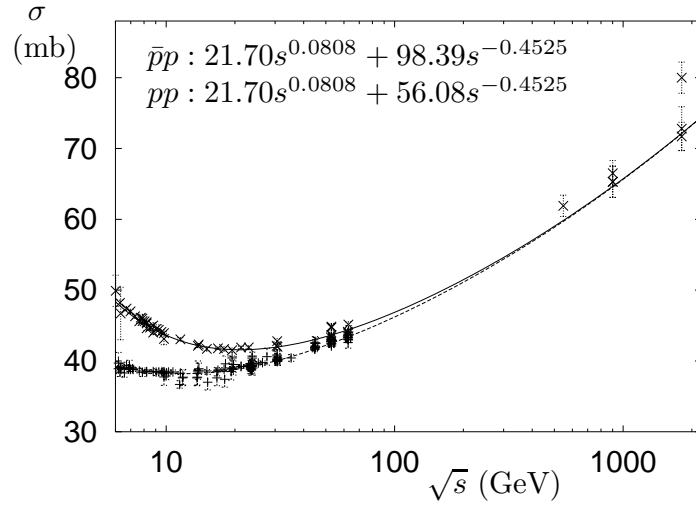


Figure 2: Fit for all accelerator data on the total  $pp$  and  $p\bar{p}$  cross sections available before LHC [4]. The curves asymptotically grow as a power of energy.

are much more precise. Therefore, one can hope that the LHC data, especially at its maximal energy (not reached yet), may be able to clarify the situation.

It is interesting (and useful), however, to examine also the theoretical basis of the Froissart theorem. This was just the aim of the paper [5], which revises derivation of the theorem and meaning of its results. The paper may be easily reached either in the journal or as the arXiv e-print, so it is not necessary to present here all its calculations and formulas. Instead, it is sufficient to describe the main results and conclusions of the paper.

- A necessary *physical* input for the Froissart theorem is, of course, unitarity. It works in two ways: on one side, the scattering-channel unitarity restricts elastic partial-wave amplitudes; on the other side, the cross-channel unitarity relates positions for scattering angle singularities of the elastic amplitude with the mass spectrum in the cross channel.
- Another *physical* input is the absence of massless particles. It guaranties the absence of angle singularities both inside the physical region and on its edges.
- A necessary *mathematical* base for the Froissart theorem is provided by properties of the Legendre functions. Especially important appears the behavior of  $P_l(z)$  at  $l \rightarrow +\infty$ . The infinite point in the  $l$ -plane is an essential singularity for the Legendre functions. As a result, their asymptotic forms at large positive  $l$  are sharply different in the three cases: inside the  $z$ -interval  $(-1, +1)$ , at its edges (*i.e.*, at  $z = \pm 1$ ), and outside this interval, though the points  $z = \pm 1$  are not singular for  $P_l(z)$  with physical (integer positive) values of  $l$ . On one side, therefore, discontinuities become possible (and arise indeed) between high-energy asymptotics of an elastic amplitude in the three configurations: inside the physical region of angles, at its boundary (*i.e.*, for the forward or backward scattering), at nonphysical (complex) angles (it is worth to emphasize that transitions between those three configurations do not touch any singularities of the amplitude). On the other side, due to properties of  $P_l(z)$ , the rate of high-energy increase of the amplitude is much more moderate for physical angles than for nonphysical ones. Such sharp moderateness of the amplitudes in physical configurations is just *the true meaning* of the Froissart theorem.
- All those results do not fix, however, any particular asymptotic expression for the total cross sections. To obtain the familiar “canonical” re-

striction of the form  $\ln^2 s$ , one should add the hypothesis that in every nonphysical configuration (even including arbitrary nonphysical angles) the amplitude cannot grow with energy faster than some finite power of energy. The Froissart paper [2] “hides” this hypothesis in dispersion relations with a finite number of subtractions. Note that no physical or mathematical justifications have been ever suggested for such an asymptotic hypothesis. Moreover, the observed linearity of Regge trajectories provides phenomenological arguments against the power boundedness (more detailed motivation see in Ref.[5]). In a general case, the upper bound for the total cross section may grow with energy approximately as the squared logarithm of the fastest asymptotics of the amplitude in nonphysical configurations.

- The more exact asymptotic expressions for the Legendre functions, used in Ref.[5], allowed to strengthen the original Froissart inequalities [2] for physical amplitudes (and cross sections). For example, even if the amplitude is bounded by a finite power of energy in any nonphysical configurations, the corresponding total cross section still *cannot* grow as  $\ln^2(s/s_0)$  with a fixed scale  $s_0$  (as is usually stated). Instead, the scale  $s_0$  itself should grow logarithmically with energy, reducing the growth rate for the total cross section.
- Increase of a total cross section faster than the log-squared energy does not mean violation of unitarity and is *not forbidden* by any general principles, contrary to a widespread opinion.
- It is interesting that neither dispersion relations, nor any particular properties of interactions were needed in the analysis of Ref.[5]. The strong interactions, as an object to apply Froissart restrictions, are marked out only by the fact of absence of massless hadrons (as compared, say, to the electrodynamics with its massless photon).

LHC has begun to contribute into the problem of the increasing total cross sections. The recent analysis of accelerator data for  $pp$  and  $p\bar{p}$  scattering [6] assumed the asymptotic behavior of their total cross sections in the form  $(\ln s)^\alpha$ , the exponent  $\alpha$  being a free parameter. The earlier data agree with the “canonical” value  $\alpha = 2$ . However, addition of the first LHC data [7] appears to provide small but statistically meaningful excess  $\alpha > 2$  [6].

Approach of Ref.[5] enables one to investigate the high-energy asymptotics not only at a fixed scattering angle (as in Ref.[2]), but also at a fixed momentum

transfer. This allows to study asymptotics of the diffraction peak slope as well. As appears, if the total cross section increases with energy, then the diffraction slope should increase at the same rate or even faster. In the saturation regime, when the total cross section grows with the maximal possible rate, its ratio to the slope should stay constant or even decrease [5]. Such expectation was in agreement with the pre-LHC accelerator data, but LHC seems to violate it [8]. This means that the present increase of the total cross sections is not saturated yet, and when going to even higher energies we may encounter some unexpected features.

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